Monotonicity Analysis over Chains and Curves

Dan Kucerovsky

University of New Brunswick, Canada

Daniel Lemire (Presenting!)

University of Quebec at Montreal (UQAM), Canada

Why care about monotonicity?

Monotonicity is:

- ▷ morphologic: slopes don't matter, you can "time warp"!
- ▷ sample-removal oblivious: missing points don't matter.

Our motivations

- ▷ Curve/Chain reconstruction for unordered data points: "best" chain to tie up data points?
- ▷ Data Mining: discover frequent patterns.
- ▷ Optical Character Recognition.
- ▷ (Motion) Signal Processing: noise removal, etc.
- ▷ High Frequency Trading, Autonomy Architecture (robotics), Automated Monitoring, Decision Support

What are chains?

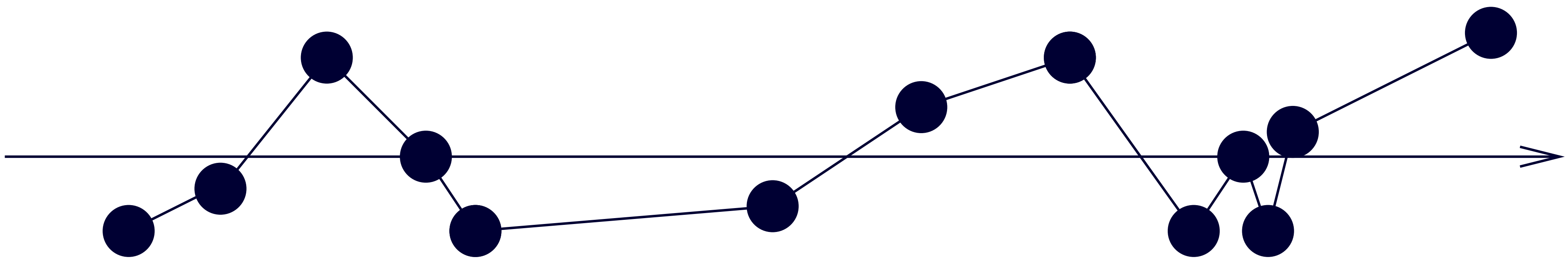
- ▷ chain: a discretely sampled curve.
- ▷ for our purposes: finite and ordered set of data points in $\mathbb{R}^d$.

What is a monotone function?

Proposition 1. *$f : \mathbb{R} \to \mathbb{R}$ is monotone if and only if $f^{-1}(B)$ is connected for all balls B.*

Monotone Chains?

A chain is x-monotone if the data points $(x_i, y_i, \ldots)$ are sorted in order of increasing x_i's [AHPMW02, DBGH00, HS91].



Extending Monotonicity!

The definition of monotonicity has been extended to real-valued functions [Bro04, CSA00, CL03, HHVW96, Mor69, vKvOB+97] ($f : \mathbb{R}^n \to \mathbb{R}$): $f^{-1}([a,b])$ connected.

Chain appraisal through curve fitting

- ▷ we could measure the “smoothness” of a chain by measuring how closely one can fit it to a smooth curve [GA94]
- ▷ BUT want chains to be **first-class citizens**!

Show first it makes sense with curves

A sensible definition for chains ought to make sense for general curves.

What is a monotone curve?

Definition 1. *An arc-length parametrized curve $s : t \to s(t)$ is R-monotone for $R > 0$ if the inverse image of any closed ball of radius at most R, under s, is connected.*

Note: straight lines are R-monotone for all $R > 0$.

Monotone curves are not continuous

- ▷ Monotone functions don't have to be continuous.
- ▷ Neither do monotone curves: – – –

Monotone curves are diff. a.e.

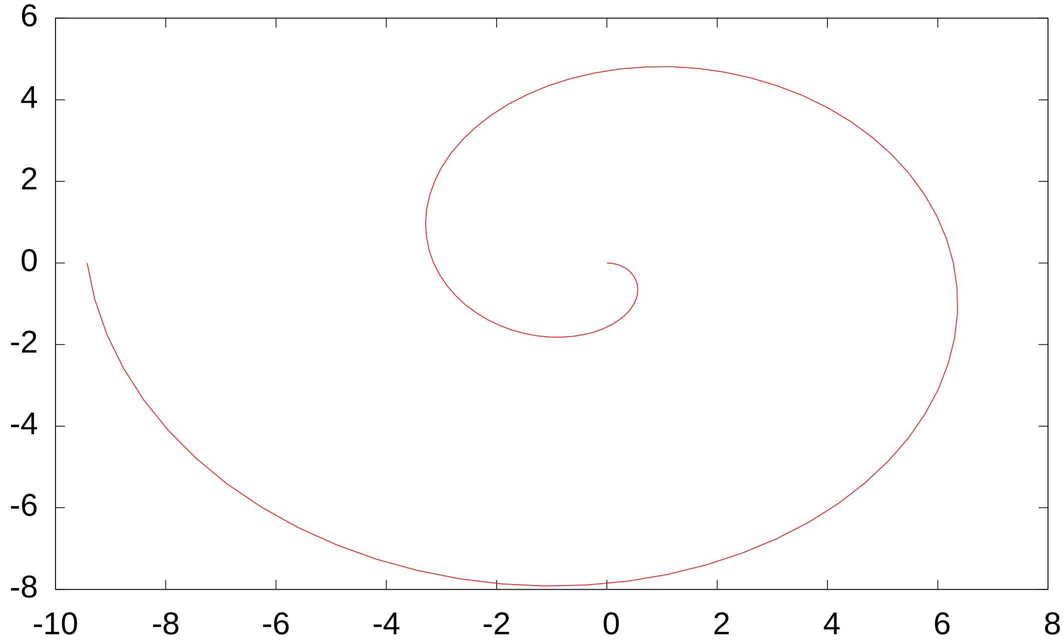
Monotone functions are diff. a.e.

Proposition 2. *Continuous R-monotone curves are differentiable a.e.*

But not twice diff.!

Differentiable curves are not R-monotone for any $R > 0$

Differentiable functions are not necessarily monotone.



Monotonicity is a predicate!

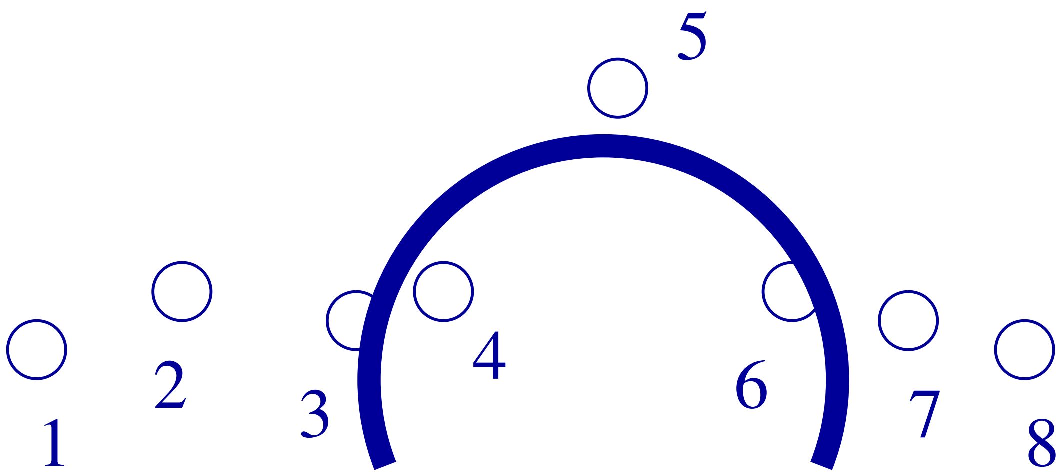
Functions are monotone or not, and there is no “degree of monotonicity.” Similarly, for curves of finite length, it simply matters whether they are R-monotone for some finite R since R-monotonicity is scale-dependent.

Proposition 3. *Given a R-monotone curve, scaling the curve by a factor $\infty > \alpha > 0$ makes it αR-monotone.*

Definition of a monotone chain

Definition 2. *A vector-valued signal* $\mathbf{p}_i$ *has a degree of monotonicity* R *if* R *is the largest value such that, for any 3 consecutive samples,* $\mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{p}_{i+2}$*, the index set of the values contained in any closed ball* B *of radius at most* R *is an uninterrupted sequence.*

Definition of a monotone chain



Motivating the definition

Proposition 4. *A (unidimensional) signal p_i is monotone if and only if, for any 3 consecutive samples, p_i, p_{i+1}, p_{i+2}, the index set of the values contained in any closed interval $[a, b]$ is a set of consecutive integers $[j, k]$. Equivalently, the index set is a convex set under an appropriate definition of convexity.*

Why care about the degree of monotonicity (R)?

- ▷ Noise tends to reduce R by creating sharp turns and local back-tracking and a highly monotone curve (R large) is more likely to be noise-free.

- ▷ When reconstructing chains from unordered sets of points, as happens in computer vision, we often want to minimize sharp turns and backtracking.

Chain Monotonicity is robust

Proposition 5. *If one point is omitted from a vector-valued signal, the degree of monotonicity cannot decrease.*

Computing R- Heron's formula

Given $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^n$, thee radius of the circle passing through them (denoted $\frown (\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)$) is where:

$$R^{\mathrm{outcircle}} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where $a = \|\mathbf{p}_1 - \mathbf{p}_2\|$, $b = \|\mathbf{p}_2 - \mathbf{p}_3\|$, $c = \|\mathbf{p}_1 - \mathbf{p}_3\|$, $s = (a+b+c)/2$.

Computing R- Putting it together

Theorem 1. *The degree of monotonicity for the sequence* $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ *is*

$$R := \begin{cases} \frac{1}{2}c & \text{if } a^2 + b^2 > c^2 \\ R^{\text{outcircle}} & \text{otherwise} \end{cases}$$

Computing R

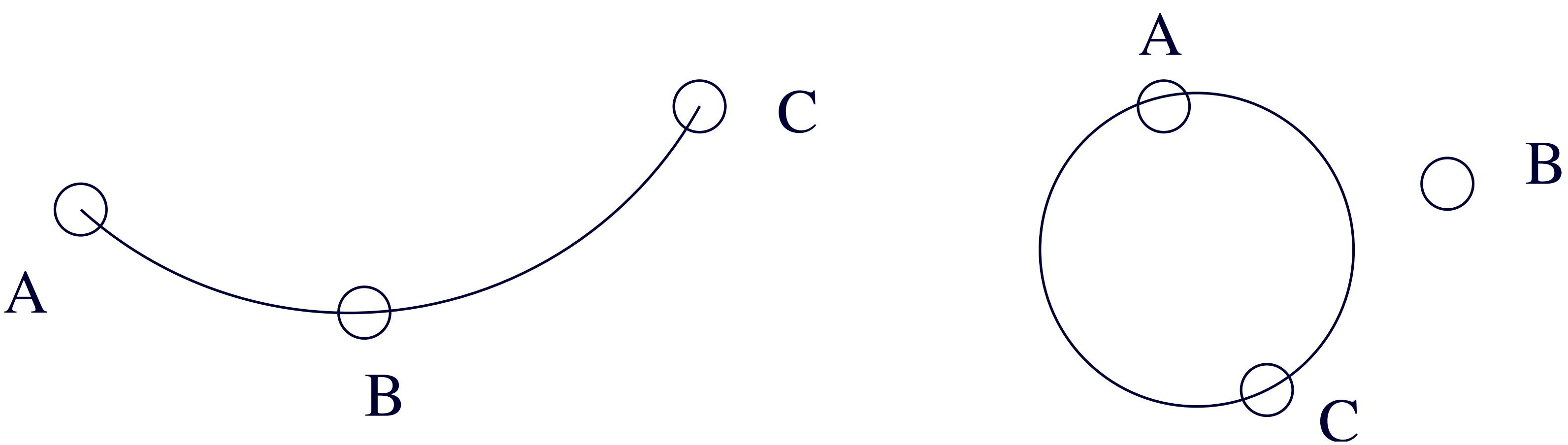


Figure 1: Given a chain of 3 data points, we give two cases: (left) the angle $\angle(ABC) > \pi/2$ so we compute the radius of the circle going through ABC, otherwise (right), we compute half the distance between A and C.

Sphere Preserving Filters

A filter is sphere-preserving (resp. circle-preserving) if, when the input data points are on a sphere (resp. circle), the filtered data points also lie on the same sphere (resp. circle). No linear filter except the identity can be sphere-preserving (SP) or circle-preserving (CP).

Sphere Preserving Filters

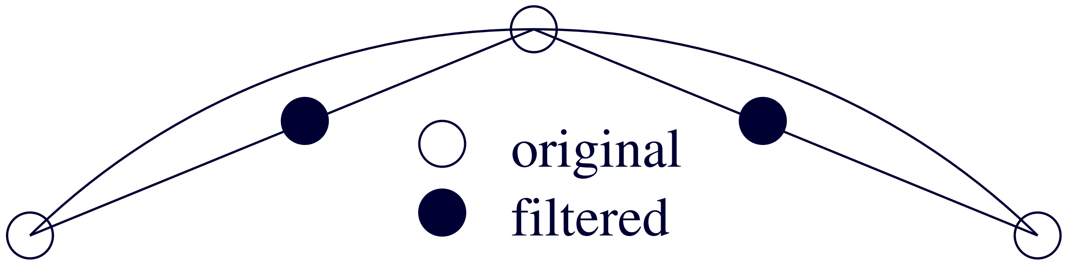


Figure 2: Given samples on circle, a simple moving average does not preserve the embedding.

Filter Design

Proposition 6. *Given* $X' = f(A,B,X,C,D)$, *if* f *is such that the degree of monotonicity*

$$R(A,B,X',C,D) \geq R(A,B,X,C,D),$$

then the recursive filter

$$\mathbf{p}_i' = f(\mathbf{p}_{i-2}', \mathbf{p}_{i-1}', \mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{p}_{i+1})$$

never decreases the degree of monotonicity of a signal.

Filter Design – stupid result

Proposition 7. *$R(A,B,X',C,D)$ is minimized when $X' = B$ or $X' = C$ and these choices are unique unless $\frown(ABC) = \frown(BCD)$ in which case any point on the arc of the circle between B and C inclusively qualifies.*

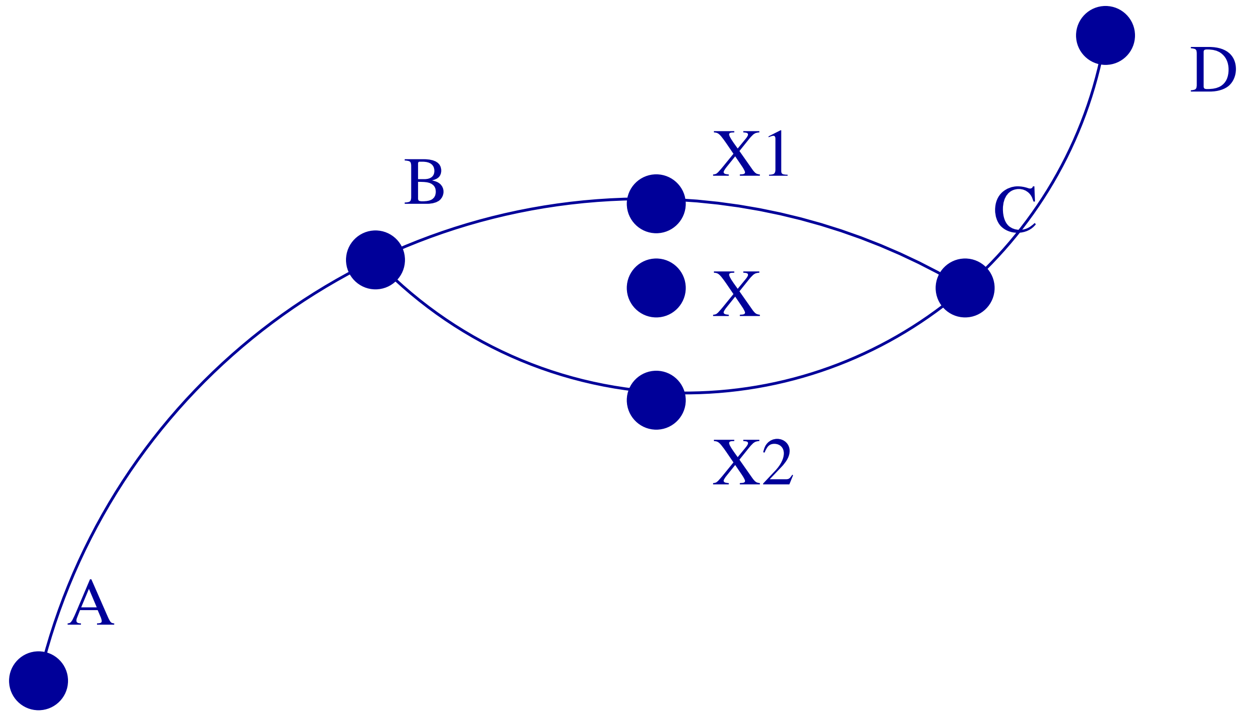
Filter Design – more interesting

Let us define X_1 to be the projection of X on the arc BC of the circle ABC,and define X_2 to be the projection of X on the arc BC of the circle BCD. Intuitively, either point X_1 or X_2 would make a good choice for X' . To ensure that the degree of monotonicity is never decreased, we set

$$f(A,B,X,C,D) = \arg\max_{X' \in \{X, X_1, X_2\}} R(A,B,X,C,D).$$

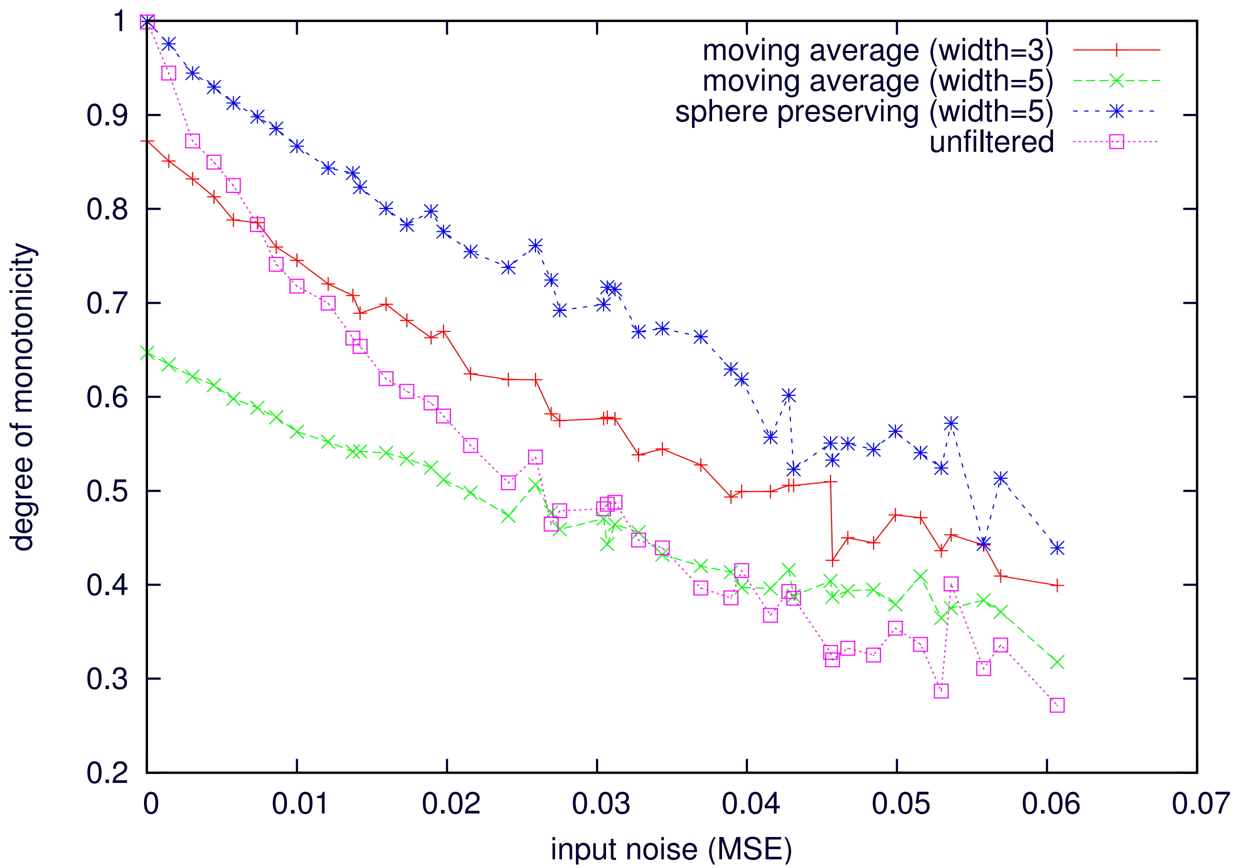
This function can be computed quickly and is sphere-preserving.

Filter Design – more interesting

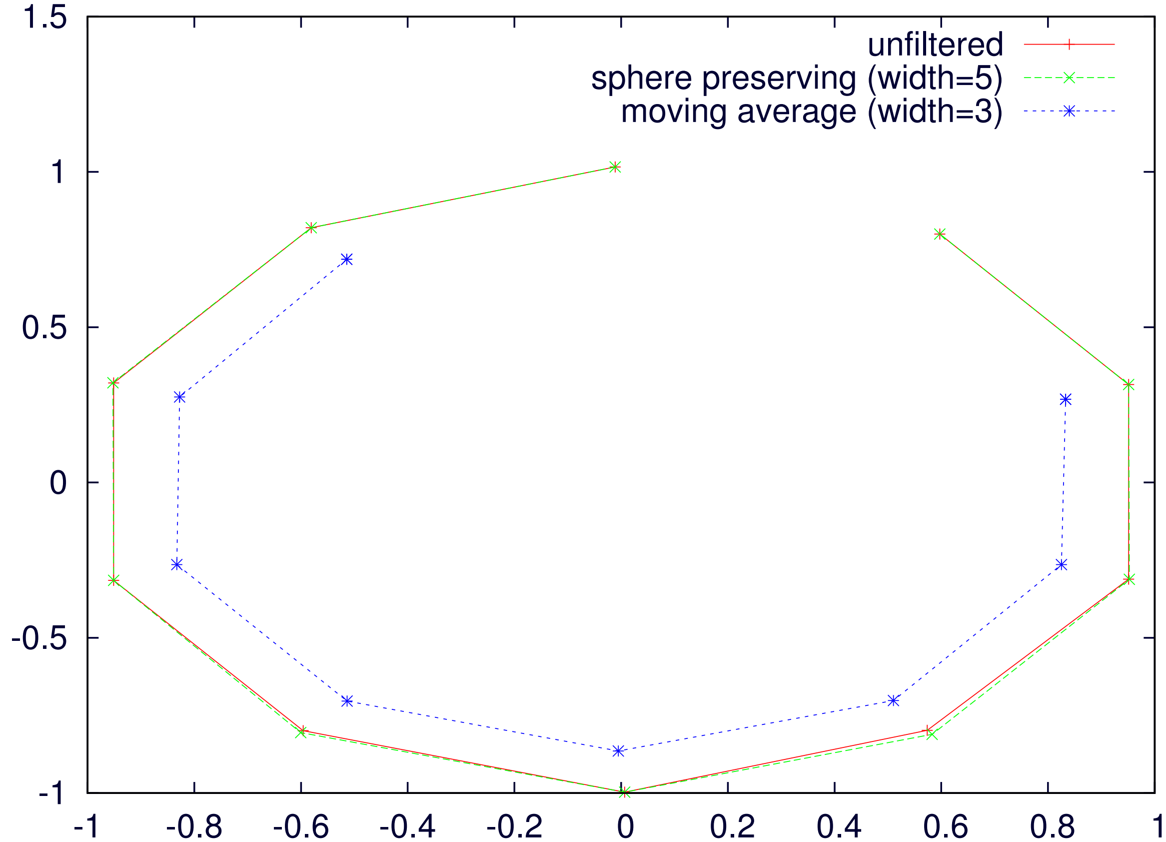


Synthetic data

Chain in the xy plane by regularly sampling a unit circle 3 times for a total of 30 samples. We add white noise.



Sample result



Conclusion

▷ a method for increasing the degree of monotonicity would also function as a good noise reduction technique.

▷ open problem: given set of unordered points, solve for the best chain minimizing monotonicity

The End!

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